Math 156 Applied Honors Calculus II Fall 2017

hw10 , due: Wednesday, November 29

1. The age of organic remains can be determined by radiocarbon dating as follows. When cosmic rays enter the atmosphere, they convert nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of 5730 years. Living animals absorb ^{14}C through the atmosphere and food chain, and when they die, the ^{14}C present in the remains decreases by radioactive decay. Consider the case of parchment, a thin material made from animal skins used for writing in ancient times. A parchment fragment was discovered having 74% as much ^{14}C as in current living animal skins. Find the age of the fragment.

2. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$ two ways.

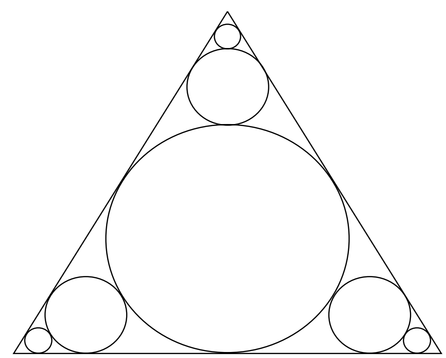
method a) Express $\dfrac{1}{n(n+1)}$ by partial fractions; examine the partial sums s_n of the series.

method b) Sketch the curves $y = x^n$ for $0 \le x \le 1$ and $n = 0, 1, 2, 3, 4$ on a common plot; let A_n be the area between $y = x^n$ and $y = x^{n+1}$ for any $n \ge 0$; compute A_n in terms of n; express the area of the unit square in terms of the A_n; deduce the sum of the series.

3. Find the interval of convergence and sum of the power series.

a) $\displaystyle\sum_{n=0}^{\infty} (x-4)^n$ b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{3^n}$

4. Three infinite sequences of tangent circles approach the vertices of an equilateral triangle. The figure shows the first few circles. Assume the triangle has sides of length 1. Express the area covered by the circles as a series. What fraction of the triangle area is covered by the circles?



5. This problem continues the discussion of Taylor approximation.

a) Recall from hw9: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2} f'''(t)\,dt$.

Now show that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$.
(hint: in the result from hw9, set $u = f'''(t)$, $dv = \frac{(x-t)^2}{2}dt$, and integrate by parts)

b) Define $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$. Note that $T_3(x)$ is a cubic function of x; it is called the Taylor polynomial of degree 3 for $f(x)$ at $x = a$. Show that $T_3(x)$ and $f(x)$ have the same function value, 1st derivative value, 2nd derivative value, and 3rd derivative value at $x = a$.

c) We view $T_3(x)$ as a cubic approximation to $f(x)$. Show that the error satisfies $|f(x) - T_3(x)| \le \frac{1}{4!} M_4 |x-a|^4$, where $M_4 = \max|f^{(4)}(t)|$. (hint: part (a) implies that $f(x) = T_3(x) + \int_a^x \frac{(x-t)^3}{3} f^{(4)}(t)\,dt$)

d) Let $f(x) = e^x$, $a = 0$. Find $T_3(x)$. Sketch $f(x), T_3(x)$ on the same graph around $x = a$.

e) Make a table with the following format. column 1: $|x-a|$, column 2: $|f(x) - T_3(x)|$. Take $f(x) = e^x, a = 0$ and fill in the entries for $x = 1, 1/2, 1/4, 1/8$ using a calculator. When $|x-a|$ is reduced by a factor of $\frac{1}{2}$, by what factor is the error $|f(x) - T_3(x)|$ reduced?