Math 156 Review Sheet for 1st Midterm Exam Fall 2018

For full credit, justify your answer, and give the units if appropriate. You may use the following integrals, but all others should be derived.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \; (n \neq -1) \; , \; \int \frac{dx}{x} = \ln x \; , \; \int e^x dx = e^x \; , \; \int \sin x dx = -\cos x \; , \; \int \cos x dx = \sin x$$

$$\int \sec x dx = \ln(\sec x + \tan x) \; , \; \int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln(\sec x + \tan x))$$

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1. True or False. Justify your answer with a reason or counterexample.

a) $\sum_{i=1}^{12} 2i = 156$ b) $\sum_{i=1}^{12}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{12}{13}$ c) $\sum_{i=0}^{n}(n-i)^2 = \sum_{i=0}^{n} i^2$ d) $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$

e) $1+3+5+7+\cdots+(2n-1) = n^2$ f) If $\int_a^b f(x)dx > 0$, then $f(x) > 0$ for $a \leq x \leq b$.

g) If an integral $\int_a^b f(x)\,dx$ is computed using the right-hand Riemann sum and the number of intervals n is doubled, then the error is approximately also doubled.

h) $\frac{d}{dx}\int_0^{x^3} \sqrt{1+t^2}\,dt = \sqrt{1+x^6}$ i) $\int_0^\infty e^{-x}\cos x\; dx = \int_0^\infty e^{-x}\sin x\; dx$

j) A spring has natural length 10 cm. If 2 J of work are needed to stretch the spring from length 10 cm to 15 cm, then 8 J of work are needed to stretch it from length 10 cm to 20 cm.

k) A cable hanging from the top of a tall building has length L m, uniform cross-sectional area A m^2, and density ρ kg/m^3. The cable is pulled to the top of the building. If the length of the cable is doubled, then the work also doubled.

l) If $\lim_{x\to\infty} f(x) = 0$, then the improper integral $\int_1^\infty f(x)\,dx$ converges.

m) The area under the graph of $y = \frac{1}{x^2}$ from $x = 1$ to $x = \infty$ is finite.

n) If $0 \leq f(x) \leq g(x)$ for $x \geq 1$ and $\int_1^\infty g(x)dx$ converges, then $\int_1^\infty f(x)dx$ also converges.

o) The error function, defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, satisfies $\text{erf}''(0) = \text{erf}(0)$.

p) $\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^2 x dx$

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section 1.2 area, 1.3 definite integral, 1.4 FTC

2. Express the integral as a limit of Riemann sums, evaluate the limit, and check by the FTC.

a) $\int_0^2 x dx$ b) $\int_0^1 x^3 dx$ c) $\int_a^b x^2 dx$ d) $\int_0^1 e^{-x} dx$

3. Evaluate the limit by any means.

a) $\lim_{x\to\infty} xe^{-x}$ b) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n}$ c) $\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\,dt$ d) $\lim_{x\to 0}\frac{e^x - 1}{x}$ e) $\lim_{r\to 1}\frac{1-r^{11}}{1-r}$

4. Find the antiderivative.

a) $\int xe^{-x^2}dx$ b) $\int x^2 e^{-x} dx$ c) $\int x\sin x dx$ d) $\int \frac{dx}{4-x^2}$ e) $\int \frac{dx}{\sqrt{4-x^2}}$ f) $\int \sqrt{4-x^2}\,dx$

5. Prove. a) $\frac{1}{20} \leq \int_0^1 \frac{x^9}{1+x}dx \leq \frac{1}{10}$ b) $\int_0^1 x\,(1-x)^{11}dx = \frac{1}{156}$

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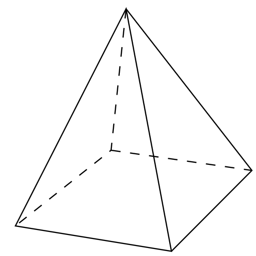
section 1.5 work

6. A force of 30 N is needed to stretch a spring from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

7. Two ions of charge q repel each other with force $f(r) = -\dfrac{q^2}{4\pi\epsilon_0 r^2}$ N, where ϵ_0 is the vacuum permittivity and r is the distance between the ions measured in meters. If one ion is held fixed at $x = 0$ mm, find the work done in moving the second ion from $x = 3$ mm to $x = 2$ mm.

8. A pyramid is built of stone with density $\rho\,\mathrm{kg/m^3}$. The base of the pyramid is a square, and the vertex is directly above the center of the base. The length of a side of the base is L m and the height of the vertex above the base is H m. a) Derive a formula for the work done in building the pyramid (i.e. raising the stone from ground level to its level in the pyramid). b) If the length L and height H are doubled, by what factor does the work increase? c) Which requires more work, building the lower half or the upper half of the pyramid?



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section 1.6 improper integrals

9. Determine whether the integral converges or diverges. If it converges, find the value. If it diverges, give a reason.

a) $\displaystyle\int_1^\infty \frac{dx}{x^4}$ b) $\displaystyle\int_0^\infty x^2 e^{-x} dx$ c) $\displaystyle\int_0^\infty e^{-x}\sin x dx$ d) $\displaystyle\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$ e) $\displaystyle\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$

f) $\displaystyle\int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}}$ g) $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ h) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$ i) $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^2}} dx$ j) $\displaystyle\int_1^\infty \frac{dx}{x^2-1}$

k) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ l) $\displaystyle\int_0^1 \frac{dx}{x^{3/2}}$ m) $\displaystyle\int_0^1 \frac{dx}{1-x}$ n) $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2} dx$ (hint: substitute $u = x^{-1}$)

10. A patient receives an intravenous drug at the rate $r(t) = 2te^{-2t}$ ml/sec, where t is the time in seconds since the treatment started. (a) Find the total dose the patient receives in the limit $t \to \infty$. (b) What fraction of the total dose is received in the first 5 seconds?

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section 1.7 arclength

11. Find the arclength of the curve on the interval $0 \le x \le 1$.

a) $y = \sqrt{1-x^2}$ b) $y = \int_0^x \sqrt{1-t^2}\, dt$ c) $y = \dfrac{e^x + e^{-x}}{2}$ d) $y = \sqrt{x^3}$ e) $y = 2x^2$

12. Sketch the curve $y = \sqrt{2x - x^2}$ for $0 \le x \le 2$ and find its arclength.

13. Sketch the curve $y = \sqrt{x}$ for $0 \le x \le 1$ and find its arclength. (hint: substitute $y = \sqrt{x}$ in the arclength integral)

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miscellaneous

14. Sketch the graph of each function on the interval $0 \le x \le 2\pi$. What do you notice about (d) and (e)? a) $\cos x$ b) $\cos 2x$ c) $\frac{1}{2}\cos 2x$ d) $\frac{1}{2} + \frac{1}{2}\cos 2x$ e) $\cos^2 x$

15. The <u>average value</u> of a function $f(x)$ on the interval $a \le x \le b$ is $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$, and the <u>root mean square value</u> is $f_{\text{rms}} = \sqrt{\dfrac{1}{b-a}\displaystyle\int_a^b (f(x))^2 dx}$. Find f_{avg} and f_{rms} for $f(x) = \cos x$ on the interval $0 \le x \le 2\pi$. (hint: do problem 14 first)